TOPOLOGY, ALGEBRA, DIAGRAMS
Barcelona December 2009

A topological space? One visualizes curves, discs, circles, mobius strip, the surface of a sphere, a torus, spheres with holes, and knots inside the three familiar dimensions of Euclidean space. Topology studies continuous deformations – stretching, twisting, bending, and so on, but not tearing -- of these spaces and many others. As such, it offers mathematical models of continuous change -- analogue variation -- in contrast to the discontinuities of digital models.

Mathematical objects or structures are never isolated. They belong always to types or species of objects to which they are structurally akin and they come with transformations and relations between them which preserve their species-kinship. For algebraic structures, such structure-preserving functions are called **homomorphisms**. For example, if A and B are groups with an operation of +, a homomorphism f preserves the operation -- f maps x + y onto f(x) + f(y). For vector spaces structure-preserving functions are known as **linear transformations**, familiar as matrices. For topological spaces structure-preserving maps f from one space to another is ae **continuous function**: they preserve nearness -- meaning neighborhoods of an image f(x) of x are images of neighborhoods of x.

Topology in its modern algebraic sense was born in 1895 when Henri Poincaré introduced the idea of the **fundamental group** of a manifold (a special kind of topological space). Poincaré's radical innovation is the founding idea of 20th century research and the contemporary algebraic understanding of topological spaces. What did Poincaré do? And why is it so significant?

It's significance is no less than the algebraization of topology, an event analogous to the arithmetization of geometry of the 17th century, where by assigning a pair of numbers (x, y), an ordinate and co-ordinate, to each point in the plane, geometrical entities in the plane – curves -- correspond to algebraic entities – equations – which relate the x and y values of the points on the curve. The resulting two-way traffic – calculating geometrical relations and picturing algebraic ones -- made the differential calculus of curves possible.

What Poincaré did was to show how to assign a group to a topological space T by building it from classes of paths in T. To begin, one picks an arbitrary point p in T. Then one considers all the paths -- loops – which start and end at p. If a loop can be continuously deformed into another loop they are identified as members of a single class. This leaves a set of classes whose loops could not be deformed into each other. One can then define an operation + on the classes to mean composition of loops, ie, one loop followed by another to give their sum. The set of classes with this operation forms a group -- the fundamental group G(T) of T. Moreover, the assignment acts on continuous functions between spaces, sending then to homomorphisms between their fundamental groups. Its significance lies in the fact that the assignment of G(T) to T is systematic across different spaces, that is, it preserves composition of continuous functions: the image of a composite of such functions is the composite of their images. This means that topological properties and relations between spaces are imaged by algebraic properties and relations between groups.

If a space can be continuously deformed into another (coffee cup to donut), the two are homeomorphic -- topologically indistinguishable from each other. From what's been said the assignment of the fundamental group preserves this: if S is homeomorphic T then G(S) will be isomorphic to G(T). The reverse is not true. Two different – non homeomorphic -- spaces can have isomorphic fundamental groups. In other words, the group F(T) provides an algebraic image of T that is alive only to certain aspects of T. Hence mathematicians have invented other assignments of groups to spaces such as the homotopy groups (higher dimensional versions of the fundamental group) and chains of homology groups which are built from spaces of oriented triangles and their generalizations. Besides groups other kinds of algebraic structures can be assigned to topological spaces, resulting in an extraordinary rich traffic between algebra and topology.

Observe that the traffic operates according to a second-order requirement that the **assignment** of the fundamental group – which acts on functions preserving topological structure – is itself structure-preserving. In other words, algebraic topology involves a level of abstraction higher than that of the base line topological and group structures themselves.

The question arises whether the defining language of these base level entities – structures characterized and understood in terms of their internal constituents of elements and subsets – is adequate to understand the higher level phenomena that emerge from the interaction between algebra and topology.

A negative answer to this question – the recognition that a new language was needed – crystallized in the 1940s when Samuel Eilenberg, a topologist, observed to Saunders Maclane, an algebraist, that the latter's calculation of a certain algebraic structure looked identical to a calculation in topology of a well-known homology group. Out of their joint attempt to say why this might be and understand what mathematical moves were common to the two calculations, they formulated a language of what they dubbed 'categories'; a diagrammatic language of arrows and configurations of arrows that in the years following its formulation more than accomplished what they had in mind. Indeed, it has had and is still having a large-scale effect on mathematics including a re-writing of algebra and topology and a radical impact on the theory of programming as well as on mathematical logic via toposes – a fact of crucial importance, for example, to the philosophical program of Alain Badiou. To appreciate the language's novelty, the conceptual innovation of thinking in terms of arrows and diagrams, we need to look first at the language of sets which it departs from.

The mainstream picture we have (have been given by the mathematical community) of mathematics since early in the 20th century has been couched in the language of sets. By the late 19th century, Cantor's theory of transfinite ordinals and his hierarchy of infinite sets had been accepted as legitimate mathematics, but not without paradoxes, such as the set of all sets which are not members of themselves, which demanded an answer to the question: What is a set?

The response of mathematicians was to construct a system of axioms whose intended objects were sets and whose only undefined relation was 'is a member (element) of'. Confining themselves solely to the language of membership, the axioms posited the existence of certain sets – the empty set, an infinite set -- together with ways of producing new sets from existing ones as well as defining equality extensionally in terms of membership: sets A and B are equal, if every member of A is a member of B and vice versa, that is, if their interiors are identical. The axioms freed mathematics from the taint of paradox and allowed mathematicians to pursue the conceptual universalism that had made sets so attractive in the first place: namely the possibility that every mathematical entity ever conceived and every property of such an entity could be defined as a certain kind of set.

A collective of mathematicians writing under the pseudonym of Bourbaki started in the 1930s to actualize this possibility, producing over the years since then thousands of pages of rigorously re-written mathematics in which every mathematical entity is a set and every mathematical argument consists of well-formed formulas in the language of sets and the apparatus of logical quantifiers ranging over sets.

The Bourbaki enterprise successfully realizes the late 19th century foundational desire, parallel to atomism in physics, to identify the fundamental 'Dinge' of mathematics. But its authors enclose it within an extreme and puritanical interpretation of mathematical rigor according to which nothing – no notation, definition, construction, theorem, or proof -- is allowed to refer to or invoke or rely on any attribute of the physical world, not least reference to the mathematician's corporeality. An interdict which – significantly -- forbids the drawing/use of diagrams but not – how could it? -- the inscribing of symbols. Contrary to normal mathematical practice, not one of their thousands of pages, Bourbaki proudly declare, contains a single diagram.

Certain features of this set-based characterization of mathematics, besides its exclusion of diagrams, stand out.

First, objects are primary, the connections between them secondary. Although ontologically every mathematical entity whether an object (a number, group, topological space, ordered set) or a connection between objects (a function, an homomorphism, an operation) is ultimately a set, the two are not imagined to be on the same level: conceptually – epistemologically -- objects have a prior status, in that one defines a structure (a group, a space) as an underlying set together with an operation on its elements or subsets, then one considers how, as an entity, it might be related to entities other than itself.

Second, a natural accompaniment to the prioritizing of objects, is the absolute nature of their interiors. Like the sets they are based on, objects are conceived extensionally, as completely determined and characterized by their internal structure of elements: two objects are identical when their underlying sets and all the subsets determining their structure are equal.

Third, set-theory's foundational remit is inherently formalistic. Bourbaki's program of re-writing 'naïve' mathematics in a first-order logic of membership, assures its fidelity to a linear, severely abstract, disembodied logico-syntactical language and style of exposition. The program eschews the slightest hint of the un-formal. Thus, the idea behind a proof, the point of a definition, the gesture underpinning a construction, the advantage of a diagram, questions of intuition, aesthetics and motive, are all irrelevant, epiphenomenal to the formal content of a concept or the logical truth of a theorem.

Conceiving mathematics a la Bourbaki became the established norm for 'correct', 'rigorous' presentations of the subject for the better part of the 20th century. Some still hold to it. For our part its antagonism to diagrams, a casualty of how it insists on interpreting 'rigor', looms large. Thus, if one understands diagrams as pictures, visible icons of an invisible similitude, then their exclusion is explained by an iconoclastic suspicion of depiction as subversive of logic that mathematical rigor embraces. Or, leaving aside anti-visualism, if one accepts Gilles Chatelet's understanding of mathematical diagrams as not so much or not primarily as icons of similitude but as frozen gestures, then excluding diagrams is explicable under a definition of 'rigor' conceived to protect the abstract and disembodied purity of mathematics from any kind of physical or corporeal contamination.

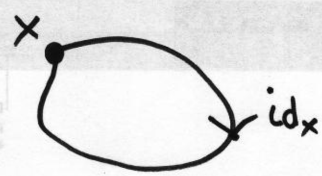
With these attributes of set theory in mind we can return to our question: what is a category?

The short answer is: a category is a bunch of objects with arrows between them obeying a few simple axioms. We have already encountered several categories. The objects can be plain old sets with functions defined on their elements as arrows. The objects can be topological spaces with arrows as continuous functions. The objects can be algebraic structures, groups for example, with arrows as homomorphisms. All manner of mathematical entities, from undifferentiated marks, graphs and knots to categories themselves can – with appropriate arrows between them -- form categories. Appropriate means the arrows obey three axioms:

Slide 1

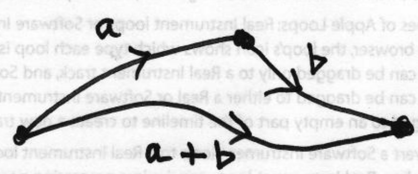
Axioms for a category
Objects and arrows added (composed) together

There is an identity arrow id_x for each object X

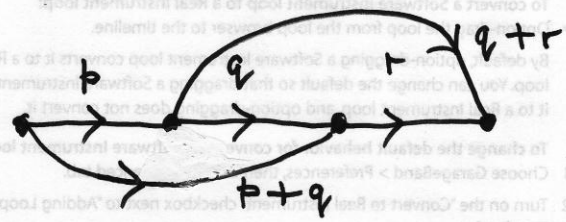


$a + id_x = id_x + a = a$ **for any arrow a**

Composition of two arrows



Associativity of three arrows



$P + (q + r) = (p + q) + r$

identity, composition, associativity axioms for a category

The class of objects of a category can be of any finite or infinite size. For example, an arbitrary bunch of sets can serve as objects of a category with an arrow x → y between sets only if x is a subset of y. And there are categories – by no means trivial -which consist of a single object with many arrows to and from itself.

As we see, the ur-concept of categories is an arrow, a morphism, that goes from one object to another. Arrows are a universal of action: configurations of them can represent/reproduce all the actions upon mathematical entities – constructions, operations, mappings, inversions, assignments, and so on – that mathematicians have devised. By taking arrows as primitive, by defining everything in terms of them, categories foreground and in turn enables a mobility of thought. In contrast to the static structuralism of set theory, categories deliver a dynamic logic – schemas of directional movement, transformation and change. By the same token, prioritizing arrows over objects works against an epistemology of interiority, against knowing/characterizing objects in terms of their internal structure. Unlike a set-based structure an object in a category is understood relationally, in terms of the arrows entering and exiting it.

The conceptual shift in mathematical discourse from interior properties to external relations central to category theory, has parallels, each of theoretical significance within its context, at a variety of sites. In fact, in mathematics itself, well before set theory's instauration, let alone the invention of categories, it is the principle governing Klein's *Erlanger Programm* of 1872 for classifying geometries from the outside as it were, in terms of their groups of symmetries. In science it becomes for Poincaré the excision of the *Ding an Sich*: "The things themselves are not what science can reach ... but only the relations between things. Outside of these relations there is no knowable reality"(1902). In linguistics, the move away from intrinsic properties to external relations is Saussure's defining structuralist turn from a referential linguistics which has 'positive terms' to one in which items are determined by their differential relations to other items. On a different terrain, it is behind the varied formulations of Vygotsky, Voloshinov and Mead which theorize the external nature of the individual 'I', as something socially and not endogenously produced; a move which reappears in Lacan's psychoanalytic understanding of the unconscious as something determined externally "as the discourse of the other". And in terms of the evolution of cognition, it captures Merlin Donald's move away from the myth of the isolated mind towards thought facilitated through external medialogical apparatuses. In short, category theory, with several significant precedents and parallels, thinks from the outside-in and not from the inside-out – thinks socio-culturally as it were – about mathematical objects.

Arrows operate on three distinct but interconnected morphic levels:

arrows between objects -- *categories*

arrows between categories – *functors*

arrows between functors -- *natural transformations*

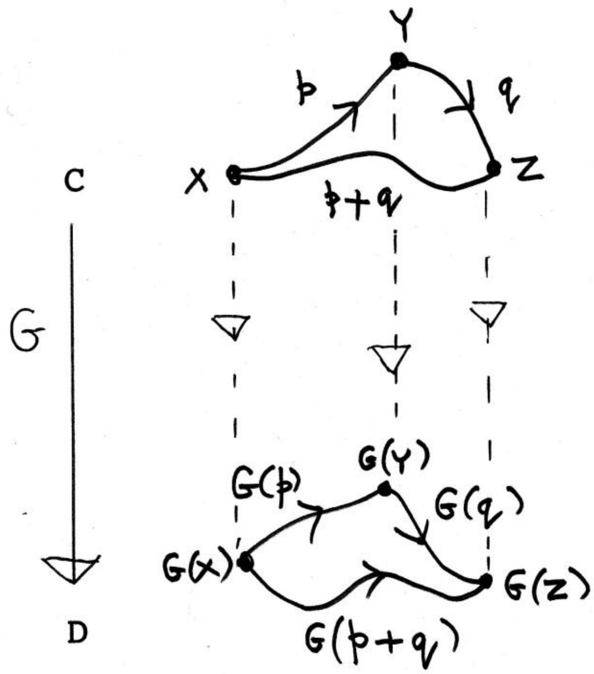
Let me turn to the second morphic level here -- the idea of a functor as an arrow, a morphism, between categories. Functors permeate mathematics. They are in essence structure-preserving functions between categories. One might think of them as providing metaphors or images of their source category. We have already encountered an important example, a functor from the category of topological spaces to that of groups, namely Poincaré's assignment of the fundamental group to a topological space.

A functor G from category **C** to category **D** is a double mapping:

Slide 2 (see next page)

Functor G from category C to category D

G preserves the composition of arrows: the G-image of the composite is the composite of G-images



$$G(p + q) = G(p) + G(q)$$

diagram of functor G:C → D between categories

First: G maps objects X and Y in **C** to objects G(X) and G(Y) in **D**.
Second: G maps arrows f: X → Y in **C** to arrows G(f): G(X) → G(y) in **D**. And it does this so that composition of arrows is preserved: the transform of a composite is the composite of transforms.

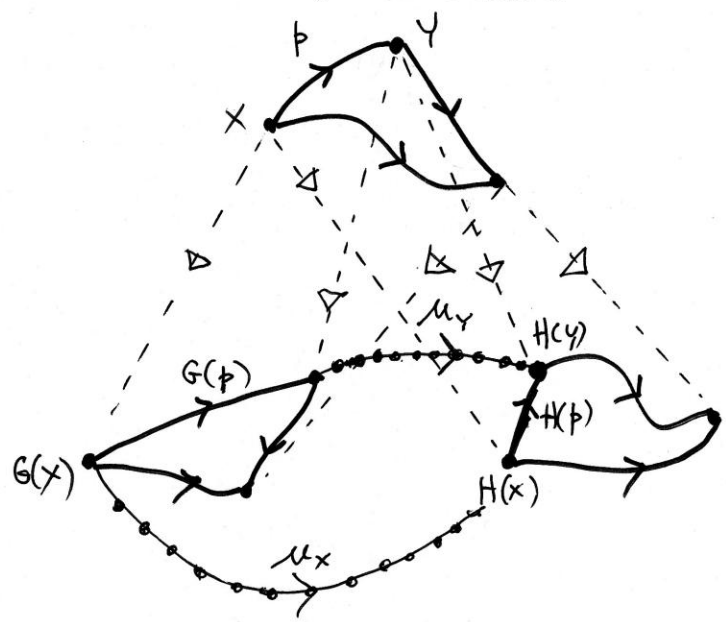
In a sense, then, functorial connections are the category version of metaphors, in that they transport aspects of a category into new settings. Observe that the conditions functors satisfy are precisely what according to the axioms arrows have to satisfy to be the morphisms inside a category of categories.

Finally, the third morphic level – that of a natural transformation. This is an arrow – a morphism – from one functor to another in which functors are re-imagined as objects.

Slide 3 (see next page)

Slide 3

Natural Transformation μ:G → H between functors is a family of arrows μ_X for X in C between the images of G and H



The two paths from G(X) to H(Y) are equal:

$$G(p) + \mu_Y = \mu_X + H(p)$$

That is, the diagram

$$\begin{array}{ccc} G(X) & \xrightarrow{\mu_X} & H(X) \\ {\scriptstyle G(p)}\downarrow & & \downarrow{\scriptstyle H(p)} \\ G(Y) & \xrightarrow[\mu_Y]{} & H(Y) \end{array}$$

Commutes for all X and Y in C

diagram of a natural transformation μ: G → H between functors G and H as a volley of arrows

It was precisely in order to formulate the idea of a natural transformation that Eilenberg and Maclane invented categories.

Functorial connections pervade mathematics no less than metaphors pervade speech and there are generally many functors between two categories. For example, there are different ways of assigning a group to a topological space, and one wants to be able to compare them to each other. In category terms, this means creating an appropriate arrow between them. The concept of a natural transformations accomplishes this.

Comparing functors means comparing their different effects on the same objects and arrows. So, if G and H are functors from **C** to **D** we need to compare their images, to compare G(X) with H(X) and G(Y) with H(Y) as X and Y range over **C**. We can only do this via arrows in **D** – μ_X -- from G(X) to H(X) and μ_Y from G(Y) to H(Y). But G and H also assign arrows in **D** to those in **C** and the two assignments have to be compatible.

A particularly valuable use of natural transformations occurs when the functors G and H go in opposite directions. For example, G might be the fundamental group functor assigning a group to a manifold and H a functor which goes in the reverse direction, assigning a manifold to a group. In general such functors are not necessarily opposites or inverses of each other. However, they frequently exhibit a relation of partial or conceptual inversion, a morphically graded refinement of complete reversal. Using a pair of natural transformations one can capture such an idea via the ubiquitous and all-important concept of two functors being **adjoint**.

But I've said enough, I think, to indicate that category's apparatus of arrows and diagrams constitutes a subtle, intricate, mathematically fecund and wide-ranging language; it's grammar of transformations and its multi-level syntax of arrows from one object, category, functor to another allow mathematics to be understood and practiced as diagrammatic thought. As well as a language it is also a species of structuralism. But not in the dominant mid 20th century sense of that term as a system of binary oppositions between immutable 'structures' (Jakobson's labial/dental distinctive features and their uptake in Levi-Straus's raw/cooked oppositions). Not, in other words, the structuralism compatible/complicit with the set-theoretic thinking of that period, a structuralism founded on an ontology of pure unchanging 'being' wedded to a binary logic of excluded middle. Against this, category theory offers a structuralism of becoming; it gives rise to a universe of things-in-movement, relations which are n-ary and not binary, in which everything is an outward process, an arrow. In brief, category theory puts difference before identity, relations before properties, prepositions, adverbs and verbs before adjectives and nouns. Ultimately, Heraclitus not Plato.

Let me conclude by coming back to topology and the question of a topological understanding/theorizing of cultural phenomena.

Within the field of topology Poincaré's topological spaces – **manifolds** -- are special. They are locally-Euclidean, they generalize our model of familiar space – neigbourhoods of points are homeomorphic to small regions of Euclidean space. Consequently, concepts of distance and angles and hence geometries are meaningful for them. A further specialization, a subclass of manifolds – **differentiable manifolds** – was introduced in all but name earlier by Riemann. For such spaces the connections between them are not merely continuous but <u>differentiable</u> in the sense of calculus, making them the natural arena for the concept of change -- change of structure, form, properties -- understood as motion in time, where time is conceived as a succession of instants indexed by the **continuum** – the one-dimensional differentiable manifold – of real numbers.

Just such a temporal dynamic underlies Gilles Deleuze's account of the ontogenesis of the **actual**, the material world of corporeal states of affairs, of bodies, physical things, actual processes. A dynamic that, as Manuel de Landa has made clear, finds its natural expression in the language of manifolds, phase spaces, vector fields and the like, that is, in the language of the topological representations of differential equations.

But distinct from the actual is the virtual – the ideational world of sense, structure, "incorporeal events", of endured time and latent possibilities -- that permeates life and is inextricable from all things cultural. On the other hand, there are topological spaces other than differentiable manifolds. In fact, there are spaces that do not possess a natural metric, a notion of distance between their points, let alone a metric that permits differentiation. Which means that topology in its generality offers models of continuous change that lie outside the horizons of calculus, and, indeed outside schemes of measurement and hence outside anything geometrical. Such spaces might be suitable candidates for capturing the a-metric dynamics of cultural change. Certainly, differential equations, however successful they are in modelling purely physical change, have little to say about the shape of cultural change, except perhaps as a constraint on their physicality, on their purely material actualizations as states of affairs. And much the same goes for numerical equations: one doesn't, except within quantitative forms of sociology (statistics, sociometrics), use numerical distance or employ metrical concepts as the primary means of characterizing the dynamics of **cultural** phenomena.

What this implies is that if we want to think topologically, if we want in effect to topologize cultural dynamics outside the purely numerical, we should think more broadly, more abstractly, think of topological spaces *ab initio*, as given by their mathematical definition. This entails deciding what one is to mean – in a given cultural arena -- by a neighborhood of a point in a space and what <u>*nearness*</u>

means, and hence of how a continuous transformation is essentially a function which preserves nearness. The idea will perhaps be daunting, a programmatic stretch even for those already attracted to the idea of topology and convinced of its utility. But absent some such attempt to think proximity or nearness and hence capture an idea of continuity relative to a particular cultural form of interest, one is left with the default language of differential equations, phase spaces and the like, one never, in other words, escapes the physics of culture, that is, from calculus.

On this note, let me end by returning to the language of categories within which, it must be said, the organization of topological thought now takes place. It is worth asking, leaving topology aside, if categories have any purchase on cultural phenomena; whether, to put it at its most direct, thinking diagrammatically about the dynamics of cultural transformation in terms of objects and arrows is at all suggestive, whether it offers any illumination and insight. If it does, even in a fairly modest way, then one has an opening perhaps to a different temporality; a mode of thinking the dynamics of cultural change other than that tied to infinitely smooth differentiable functions and the points along the one-dimensional continuum of numbered instants that constitutes calculus -- but one articulated as a flow of arrows at different morphic levels, a scaled and diagrammatic <u>production</u> of time. But that is a topic for another occasion.